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SMOOTH BORES IN A TWO-LAYER FLUID WITH A FREE BOUNDARY

B. I. Tuleuov

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In this paper, we investigate motions of the type of a smooth bore of a two-layer fluid with a free boundary. This is a motion in which the uniform state ahead of a wave continuously goes over into the uniform state behind the wave. Following [1], we shall call the limiting states at infinity conjugate flows. Nonlinear waves in a two-layer fluid have been extensively studied over the last decades. The model of a two-layer fluid with a rigid boundary has been studied most thoroughly. In this case, the parameters of flows of the type of a bore are described in detail on the basis of the long-wave approximation [2 (Chapter 1) and 3]. The existence of such waves was confirmed experimentally and proved rigorously for exact Euler equations [5, 6]. Solitary waves in a two-layer fluid with a free boundary with no velocity shift between the layers were studied by Peters and Stoker [7]. Kakutani and Yamasaki [8] obtained solutions of the type of a bore for this case within the framework of the modified Korteweg–de Vries approximation.

In this paper, the relationships between the parameters of conjugate piecewise-constant flows with a free boundary are analyzed on the basis of the laws of conservation of mass, energy, and momentum. It is shown that, in contrast to a two-layer fluid with a rigid boundary [6], there is resonance between a bore and a linear wave: a nonlinear bore is accompanied by a periodic wave of smaller amplitude with the same phase velocity as the velocity of the bore. For admissible Froude numbers, an approximate solution in the long-wave limit describing the bore profile is constructed.

Formulation of the Problem. We consider the steady potential flow of an ideal, incompressible, heavy two-layer fluid with a free boundary above an even bottom. We assume that the heavier fluid is below. The subscript 1 denotes the quantities that characterize the flow in the lower layer, and the subscript 2 denotes the flow of the upper layer. We assume that as $x \to \pm\infty$, the flow is piecewise-constant with velocities $U_i^{\pm}$ and depths $h_i^{\pm}$ in the layers in the appropriate average sense, which will be refined below. The main dimensionless parameters are the Froude numbers

$$\mathrm{Fr}_i^{\pm} = U_i^{\pm}\sqrt{\frac{\rho_i}{\rho_1-\rho_2}\frac{1}{gh_i^{\pm}}} \qquad (i = 1,\, 2)$$

and the ratio of the depths of the unperturbed layers is $r^{\pm} = h_2^{\pm}/h_1^{\pm}$, where ρ_i is the fluid density in the corresponding layer, and $\rho_1 > \rho_2$. It is convenient to introduce the amplitude parameters $a_i = (h_i^+ - h_i^-)/h_i^-$ $(i = 1,\, 2)$ which express the relative difference in depth for each layer. In what follows, we shall use the parameters of a state that is attained as $x \to -\infty$ and omit the minus sign in the designations. The state parameters for $x \to +\infty$ are determined from the law of conservation of flow rate in the layers. We formulate the problem in a fixed range of independent variables. As the independent variables, we choose the Mises variables (x, ψ), where ψ is the stream function. We normalize ψ in such a way that the strip $0 \leqslant \psi \leqslant 1$ corresponds to the lower layer in the plane (x, ψ), and the strip $0 \leqslant \psi \leqslant 1 + r$ corresponds to the upper layer. The function describing the shape of the streamlines $y(x, \psi) = \psi + w(x, \psi)$ is the desired function. The function w should satisfy the following system of equations and boundary conditions:

$$w_{xx} + w_{\psi\psi} = \operatorname{div}\mathbf{f}(\nabla w) \quad \text{for} \quad 0 < \psi < 1 \quad \text{and} \quad 1 < \psi < 1 + r; \tag{1}$$

$$w = 0 \quad \text{for} \quad \psi = 0; \tag{2}$$

Institute of Applied Mathematics, Kazakhstan National Academy of Sciences, Karaganda 470055.
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$$(w)\Big|_{\psi=1-0} = (w)\Big|_{\psi=1+0}; \tag{3}$$

$$\mathrm{Fr}_1^2(w_\psi - f_2)\Big|_{\psi=1-0} - w(x,1) = r\mathrm{Fr}_2^2(w_\psi - f_2)\Big|_{\psi=1+0}; \tag{4}$$

$$\mu r\,\mathrm{Fr}_2^2(w_\psi - f_2) - \lambda w = 0 \quad \text{for} \quad \psi = 1 + r, \tag{5}$$

Furthermore, it should satisfy the conditions at infinity

$$w \to 0, \qquad \nabla w \to 0 \quad \text{for} \quad x \to -\infty; \tag{6}$$

$$\left.\begin{array}{ll} w(x,1) \to a_1, & w(x,1+r) \to a_1 + ra_2 \\ w_x \to 0, & w_\psi \to a_i \quad \text{in} \quad i\text{th} \quad \text{layer} \end{array}\right\} \quad \text{for} \quad x \to +\infty. \tag{7}$$

Here $\lambda = \rho_2/\rho_1$; $\mu = 1 - \lambda$; and the vector $\mathbf{f}$ has the components

$$f_1 = \frac{w_x w_\psi}{1 + w_\psi}, \qquad f_2 = \frac{1}{2}\,\frac{w_x^2 + 3w_\psi^2 + 2w_\psi^3}{(1 + w_\psi)^2}.$$

The limits in (6) and (7) will be treated as follows. It is assumed that as $x \to \pm\infty$, the solution for the lower layer is representable in the form $w(x,\psi) = a_1^\pm \psi + b w_p^\pm(x,\psi) + w_e^\pm(x,\psi)$, where $a_1^+ = a_1$; $a_1^- = 0$; $w_p^\pm$ is a periodic-in-x function with zero average over the period, b is the small-amplitude parameter of the periodic component, and the function $w_e^\pm$ tends to zero as $x \to \pm\infty$. A similar representation is implied for the upper layer. As will be seen below, the necessity of taking into account the periodic component of the solution follows from the structure of the spectrum of the linearized problem.

Analysis of the Laws of Conservation. To derive relations between the state parameters, we use the integral law of conservation of the horizontal momentum flux [2 (Chapter 1)], which in our case has the form

$$\mathrm{Fr}_1^2 \int_0^1 \frac{w_x^2 - w_\psi^2}{1 + w_\psi}\,d\psi + r\mathrm{Fr}_2^2 \int_1^{1+r} \frac{w_x^2 - w_\psi^2}{1 + w_\psi}\,d\psi = -w(x,1)^2 - \frac{\lambda}{\mu} w(x,1+r)^2. \tag{8}$$

In this integral, and also in the boundary conditions (4) and (5), which take into account the laws of conservation of mass and energy fluxes, we pass to the limit for $x \to +\infty$ with allowance for the asymptotic relation (7). Then, in the zeroth approximation for the parameter b, the Froude numbers are expressed explicitly through the amplitudes a_i:

$$\mathrm{Fr}_1^2 = \frac{2}{\mu}\,\frac{(1+a_1)^2(a_1 + \lambda r a_2)}{a_1(a_1+2)}, \quad \mathrm{Fr}_2^2 = \frac{2}{r}\,\frac{\lambda}{\mu}\,\frac{(1+a_2)^2(a_1 + r a_2)}{a_2(a_2+2)}. \tag{9}$$

The amplitudes, in turn, are related by

$$F(a_1,a_2) \equiv a_1^2(a_1 + \lambda r a_2)(a_2 + 2) + \lambda r a_2^2(a_1 + r a_2)(a_1 + 2) = 0. \tag{10}$$

In the plane of the Froude numbers (Fr_1, Fr_2), formulas (9) and (10) define the locus of points that represents admissible fluid states for $x \to -\infty$. In this case, the flow for $x \to +\infty$ has the following Froude numbers:

$$|\mathrm{Fr}_1^+| = \frac{1}{1+a_1}|\mathrm{Fr}_1|, \qquad |\mathrm{Fr}_2^+| = \frac{1}{1+a_2}|\mathrm{Fr}_2|.$$

Let us use the quantity a_1 as the main small amplitude parameter. Note that $a_1 a_2 < 0$. It is not hard to establish that $1/r < |a_2/a_1| < 1/\lambda r$, in particular, $-1 < a_1 < \lambda r$ and $-1 < a_2 < 1/r$.

Equation (10) defines the amplitude a_2 as an implicit function of a_1. For small a_1, the following expansion is valid:

$$a_2(a_1) = -\xi_0 a_1 + \xi_1 a_1^2 + o(a_1^2), \tag{11}$$

where $-\xi_0$ $(1/r < \xi_0 < 1/\lambda r)$ is a single real negative root of the cubic equation

$$\xi^3 + \frac{1}{r}\xi^2 + \frac{1}{r}\xi + \frac{1}{\lambda r^2} = 0,$$

and the coefficient ξ_1 has the form

$$\xi_1 = \frac{1}{2\lambda r}\frac{(1-\lambda r\xi_0)(1+\xi_0)}{1-2\xi_0+3r\xi_0^2}.$$

Formulas (9) with allowance for (11) lead to the relation

$$\mathrm{Fr}_2^2 = \beta_0 - \frac{1}{r^2\xi_0^2}(\mathrm{Fr}_1^2 - \alpha_0) + o(\mathrm{Fr}_1^2 - \alpha_0), \tag{12}$$

which is valid in the neighborhood of the points $\mathrm{Fr}_1^2 = \alpha_0$, $a_1 = 0$, where $\alpha_0 = (1/\mu)(1 - \lambda r\xi_0)$ and $\beta_0 = (\lambda/\mu)(1 - 1/r\xi_0)$ are the main terms of the expansion of the squares of the Froude numbers Fr_1^2 and Fr_2^2. respectively, in terms of powers of a_1.

Linear Waves. Linearization of the initial system on the piecewise-constant solution with the parameters Fr_i and r yields Eqs. (1)–(5) with $\mathbf{f} = 0$. The linearized equations have a solution in the form of periodic waves, $w_p(x,\psi) = W(\psi)e^{ikx}$, if the wave number k and the parameters of the main flow are related by the dispersion relation

$$\left(\mathrm{Fr}_1^2 - \frac{1}{\mu}\frac{\mathrm{th}\, k}{k}\right)\left(\mathrm{Fr}_2^2 - \frac{\lambda}{\mu}\frac{\tanh kr}{kr}\right) + \frac{\lambda^2}{\mu^2}\frac{\tanh k}{k}\frac{\tanh kr}{kr}\left(\frac{\mu^2}{\lambda^2}\mathrm{Fr}_2^4 k^2 r^2 - 1\right) = 0. \tag{13}$$

As $|k|$ changes from 0 to $+\infty$, the set of points $(\mathrm{Fr}_1, \mathrm{Fr}_2)$ whose coordinates satisfy (13) forms a spectrum of linear waves of infinitely small amplitude. To determine the structure of the spectrum, we treat the dispersion relation as a biquadratic equation in Fr_2. It can be factored in the form $(\mathrm{Fr}_2^2 - \mathrm{Fr}_{2/1}^2)(\mathrm{Fr}_2^2 - \mathrm{Fr}_{2/2}^2) = 0$ with the roots

$$\mathrm{Fr}_{2/1}^2 = \left(-\left(\mathrm{Fr}_1^2 - \frac{1}{\mu}\frac{\tanh k}{k}\right) + \sqrt{D}\right)\Big/(2r\tanh k\tanh kr),$$

$$\mathrm{Fr}_{2/2}^2 = \left(-2\frac{\lambda}{\mu}\frac{\tanh kr}{kr}\right)\Big/\left(-\left(\mathrm{Fr}_1^2 - \frac{1}{\mu}\frac{\tanh k}{k}\right) + \sqrt{D}\right),$$

which define families of curves that depend on the parameter k and have the discriminant

$$D = \left(\mathrm{Fr}_1^2 - \frac{1}{\mu}\frac{\tanh k}{k}\right)^2 + 4\frac{\lambda}{\mu}\frac{\tanh k}{k}\tanh^2 kr\left(\mathrm{Fr}_1^2 - \frac{1}{\mu}\frac{\tanh k}{k}\right),$$

which is rigorously positive for all finite values of k. The form of representatives of these families in the plane of the Froude numbers (Fr_1 and Fr_2) for fixed k and r is given in Fig. 1. A curve of the first family is a pair of δ-shaped curves, and a curve of the second family is an oval. Since $D \sim \mathrm{Fr}_1^2$ as $k \to \infty$, curves of the first family degenerate into the straight line $\mathrm{Fr}_{2/1} = 0$, and curves of the second family degenerate into the point $(0,0)$. Passage to the limit $k \to 0$ in Eq. (13) defines a fourth-order curve in the plane of $(\mathrm{Fr}_1, \mathrm{Fr}_2)$:

$$\left(\mathrm{Fr}_1^2 - \frac{1}{\mu}\right)\left(\mathrm{Fr}_2^2 - \frac{\lambda}{\mu}\right) - \frac{\lambda^2}{\mu^2} = 0. \tag{14}$$

This curve consists of the oval inscribed in the rectangular $[-1,1] \times [-\sqrt{\lambda}, \sqrt{\lambda}]$ and four branches of the type of hyperbolas with vertical asymptotes $\mathrm{Fr}_1 = \pm\sqrt{1/\mu}$ and horizontal asymptotes $\mathrm{Fr}_2 = \pm\sqrt{\lambda/\mu}$ (Fig. 2). Note that in [2] curve (14) is of importance for the analysis of characteristics of a quasi-linear system of equations of two-layer shallow water. The entire simply-connected region enclosed between the hyperbolic branches (which includes the oval containing curves of the second family, concentric ovals) is continuously filled with doubly skew-symmetric δ-shaped curves of the first family.

Thus, for any values of Fr_1 and Fr_2 from the indicated region, there is always a wave mode (more precisely, two wave modes propagating in opposite directions; for brevity, we shall combine them) defined by the first family; one more mode exists for Fr_1 and Fr_2, which fall within the oval (the necessary conditions

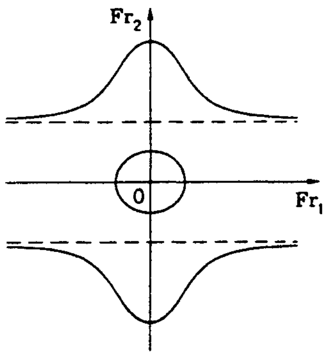

Fig. 1

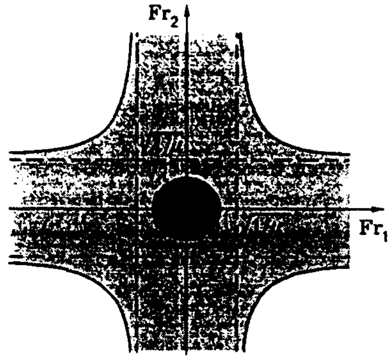

Fig. 2

are $\mathrm{Fr}_1^2 < 1$ and $\mathrm{Fr}_2^2 < \lambda$). The first of these modes characterizes fast waves with the higher amplitude on the free surface, and the second describes slow waves with the dominating amplitude on the interface. As in the case of a two-layer fluid with a rigid boundary [6], the bore type flow branches from the piecewise-constant uniform flow at the boundary point in the region of the spectrum that corresponds to the internal-wave mode.

Indeed, it is not hard to verify that the points $\mathrm{Fr}_1^2 = \alpha_0$ and $\mathrm{Fr}_2^2 = \beta_0$ belong to the oval, and it follows from expansion (12) that the curve of the possible states of bore type flows has a first-order point of tangency with the oval (one can show that the tangency is external). However, in contrast to the above-mentioned case of flows with a rigid boundary, branching occurs within the spectrum of linear waves. In other words, a bore should appear in a pair with a progressive periodic wave whose period is defined by the surface mode according to the dispersion relation (13). This situation is similar to the superposition of a solitary wave and a rapidly oscillating capillary wave on the surface of a thin fluid layer for Bond numbers $\mathrm{Bo} < 1/3$ [9].

Approximate Solution. We consider values of the parameters Fr_1^2 and Fr_2^2 that satisfy system (9) and (10). For such Froude numbers for small $a_1 \neq 0$, the dispersion relation for k has a single real root described by a curve of the first family, $\mathrm{Fr}_{2/1}$. Another root (complex) is described by a curve of the second family, and it is purely imaginary: $k = i\varepsilon$. Expansion of all quantities in the dispersion relation in power series of the parameter a_1 shows that ε and a_1 are of the same order of smallness: $\varepsilon = \varepsilon_1 a_1 + o(a_1)$ (we do not give herein the explicit expression for $\varepsilon_1 \neq 0$, since its specific form is of no significance here). The smallness of ε for small a_1 means that the flow is of a long-wave character for the corresponding Froude numbers. It is natural, therefore, to perform a transformation in the spirit of shallow water theory: $x' = \varepsilon x$ (the prime is omitted below). This is equivalent to the choice of the quantity h_1/ε as the characteristic length along the horizontal. We seek an approximate solution in the form

$$y = \psi + \sum_{n=1}^{\infty} a_1^n w_n(x, \psi).$$

Then, system (1)–(5) with allowance for the expansion of the Froude numbers gives a sequence of boundary-value problems for determining the functions w_n; for $n = 1$ we have

$$w_{1\psi\psi} = 0 \text{ for } 0 < \psi < 1 \text{ and } 1 < \psi < 1 + r, \qquad w_1 = 0 \text{ for } \psi = 0,$$

$$(w_1)\Big|_{\psi=1-0} = (w_1)\Big|_{\psi=1+0}, \quad \alpha_0(w_{1\psi})\Big|_{\psi=1-0} - r\beta_0(w_{1\psi})\Big|_{\psi=1+0} - w_1(x,1) = 0,$$

and $\mu r\beta_0 w_{1\psi} - \lambda w_1 = 0$ for $\psi = 1 + r$. For this approximation, the conditions at infinity are as follows:

$$w_1 \to 0, \quad \nabla w_1 \to 0 \quad \text{for} \quad x \to -\infty,$$

$$w_1 = \begin{cases} \psi, & 0 \leqslant \psi \leqslant 1, \\ 1 - \xi_0(\psi - 1), & 1 \leqslant \psi \leqslant 1 + r \quad \text{for} \quad x \to +\infty. \end{cases}$$

Then,

$$w_1(x,\psi) = A(x)\begin{cases} \psi, & 0 \leqslant \psi \leqslant 1, \\ 1 + (1-\psi)\xi_0, & 1 \leqslant \psi \leqslant 1 + r. \end{cases}$$

The ordinary differential equation for $A(x)$ is obtained as the condition of compatibility of equations for the third approximation: $A'' = 2A^3 - 3A^2 + A$ (the prime denotes differentiation with respect to x). The solution of the type of a bore is written with accuracy up to translation along x:

$$A(x) = \frac{1}{2}\Big(1 + \tanh\frac{\alpha x}{2}\Big).$$

Here the constant α taking values of ± 1 has the meaning of a parameter that indicates the direction of wave reversal.

In conclusion, we note that four points on the oval with the coordinates $\mathrm{Fr}_1 = \pm\sqrt{\alpha_0}$ and $\mathrm{Fr}_2 = \pm\sqrt{\beta_0}$ divide the boundary of the spectrum of linear waves of the internal mode into four arcs. Each of these arcs consists of bifurcation points at which the branches of solitary internal waves paired with linear waves of the surface mode originate.

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